MATH 2020A Tutorial 1

① $\int_0^2 \int_{-1}^1 (x-y)\,dy\,dx$

$= \int_0^2 \left[xy - \frac{1}{2}y^2\right]_{-1}^1 dx$

$= \int_0^2 \left[x - \frac{1}{2}\right] - \left[-x - \frac{1}{2}\right] dx$

$= \int_0^2 2x\,dx = x^2 \Big|_0^2 = 4$

② $\int_0^1 \int_0^1 \frac{y}{1+xy}\,dx\,dy$

$= \int_0^1 \left[\ln|1+xy|\right]_0^1 dy$

$= \int_0^1 \left[\ln(1+y) - \ln 1\right] dy$

$= \int_0^1 \ln(1+y)\,dy$

$= \ln(1+y)\cdot y \Big|_0^1 - \int_0^1 \frac{1}{1+y}\cdot y\,dy$

$= \ln 2 - \int_0^1 \frac{1+y-1}{1+y}\,dy = \ln 2 - \int_0^1 1 - \frac{1}{1+y}\,dy$

$= \ln 2 - 1 + \ln(1+y)\Big|_0^1 = 2\ln 2 - 1$

③ $\int_{-1}^2 \int_0^{\frac{\pi}{2}} y \sin x\,dx\,dy$

$= \int_{-1}^2 -y\cos x \Big|_0^{\frac{\pi}{2}} dy$ [treat y as a constant, we integrate "x"]

$$= \int_{-1}^{2} y\, dy = \frac{1}{2} y^2 \Big|_{-1}^{2} = 2 - \frac{1}{2} = \frac{3}{2}$$

④ $$\int_{-1}^{1} \int_{0}^{\frac{\pi}{2}} x \sin\sqrt{y}\, dy\, dx$$

$$= \int_{0}^{\frac{\pi}{2}} \int_{-1}^{1} x \sin\sqrt{y}\, dx\, dy$$ [by Fubini's theorem, because we don't know how to integrate $\sin\sqrt{y}$]

$$= \int_{0}^{\frac{\pi}{2}} \left[\frac{1}{2} x^2 \sin\sqrt{y} \right]_{-1}^{1} dy$$

$$= \int_{0}^{\frac{\pi}{2}} 0\, dy = 0$$

⑤ $$\int_{0}^{1} \int_{0}^{3} x e^{xy}\, dx\, dy$$

$$= \int_{0}^{3} \int_{0}^{1} x e^{xy}\, dy\, dx$$ [Fubini's theorem]

$$= \int_{0}^{3} e^{xy} \Big|_{0}^{1} dx = \int_{0}^{3} [e^{x} - 1]\, dx$$

$$= e^{x} - x \Big|_{0}^{3} = e^{3} - 3 - [1 - 0] = e^{3} - 4$$

$$\int_{0}^{1} c e^{cy}\, dy = \int_{0}^{1} e^{cy}\, d(cy)$$

⑥ In general, if

$$\int_{a}^{b}\int_{c}^{d} f(x) g(y)\, dx\, dy = \int_{a}^{b} f(x)\, dx \cdot \int_{c}^{d} g(y)\, dy$$

$$\iint_{a\,c}^{b\,d} f(x)g(y)\,dx\,dy = \int_a^b f(x)dx \cdot \int_c^d g(y)dy$$

If $\int f(x)dx = F(x)$

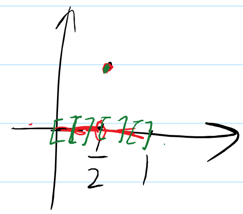
$\int g(y)dy = G(y)$.

$$L.H.S = \int_a^b F(x)g(y)\Big|_c^d dy = \int_a^b [F(d)-F(c)]g(y)dy$$

$$= [F(d)-F(c)][G(b)-G(a)]$$

$$= R.H.S.$$

⑦ Let f = f(x, y) be a bounded function defined in R = [0, 1] × [0, 1] which is 0 everywhere except at a point (1/2, 1/2). Show that f is integrable in R with integral equal to 0.

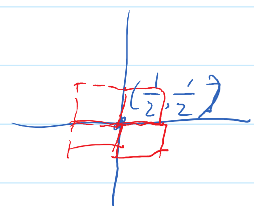


Pf: Let $P = \{(x_j, y_k): \ 0 = x_0 < x_1 < \dots < x_n = 1,$
$0 = y_0 < y_1 < \dots < y_n = 1\}$

be any partition of R.

Let $P_{jk} \in [x_{j-1}, x_j] \times [y_{k-1}, y_k]$

Then $(\frac{1}{2}, \frac{1}{2})$ lies in at most four rectangles $[x_{j-1}, x_j] \times [y_{k-1}, y_k]$



$$|S(f, \mathring{P})| = \left|\sum_{j=1}^{n}\sum_{k=1}^{n} f(P_{jk})\,\Delta x_j \Delta y_k\right|$$

$$\le 4 \cdot |f|_{max} \cdot |P|^2$$

As $|P| \to 0$, $|S(f, \mathring{P})| \to 0$

then for any $\varepsilon > 0$, there is a $\delta = \frac{1}{3\sqrt{|f|_{max}}}\sqrt{\varepsilon}$, s.t.

for any $|P| < \delta$,

$$|S(f, \mathring{P})| \le 4|f|_{max} \cdot \frac{\varepsilon}{9|f|_{max}} < \varepsilon.$$

⑧ ||

⑧

for any partition $P = \{x_j,\ 0 = x_0 < x_1 < \cdots < x_n = 1\}$

Let $P_1 \in [0, x_1]$.

$$S(f, \mathring{p}) = \sum_{j=1}^{n} f(P_j)\Delta x_j \to \infty \qquad f(P_1)\cdot\Delta x_1 = f(P_1)\cdot x_1 \to \infty$$

as $P_1 \to 0$

⑨ If two functions f and g are equal except at finitely points and lines, then the double integrals of f and g are the same.

⑩ $\int_0^1 x^{-\frac{1}{2}}\,dx \overset{\text{define}}{:=} \lim_{\varepsilon\to 0}\int_\varepsilon^1 x^{-\frac{1}{2}}\,dx.$

the Riemann integral does not exist on $[0, 1]$.

But $\int_\varepsilon^1 x^{-\frac{1}{2}}\,dx$ exists since $x^{-\frac{1}{2}}$ is continuous on $[\varepsilon, 1]$ for any $\varepsilon \in (0, 1)$

and also $\lim_{\varepsilon\to 0}\int_\varepsilon^1 x^{-\frac{1}{2}}\,dx$ exists. it is not the Riemann integration.

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② $\int_0^1 \boxed{\int_0^1 \frac{y}{1+xy}\,dx}\,dy$

$= \int_0^1 \left[\ln|1+xy|\right]_0^1 dy$

$= \int_0^1 \ln(1+y)\,dy = \ln(1+y)\cdot y\Big|_0^1 - \int_0^1 \frac{1}{1+y}\cdot y\,dy$

$= \ln 2 - \int_0^1 \frac{1+y-1}{1+y}\,dy = \ln 2 - \int_0^1 1 - \frac{1}{1+y}\,dy$

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$= \int_{-1}^2 y\,dy = \frac{1}{2}y^2\Big|_{-1}^2 = 2 - \frac{1}{2} = \frac{3}{2}$

④ $\int_1 \boxed{\int_0^{\frac{\pi}{2}} x \sin\sqrt{y}\,dy}\,dx$

④ $\displaystyle\int_{-1}^{1}\boxed{\int_0^{\frac{\pi}{2}} x\sin\sqrt{y}\,dy}\,dx$

$\displaystyle= \int_0^{\frac{\pi}{2}}\int_{-1}^{1} x\sin\sqrt{y}\,dx\,dy$ [by Fubini's Theorem]

$\displaystyle= \int_0^{\frac{\pi}{2}} \frac{1}{2}x^2\sin\sqrt{y}\Big|_{-1}^{1}\,dy$

$\displaystyle= \int_0^{\frac{\pi}{2}} \cancelto{0}{\frac{1}{2}\sin\sqrt{y} - \frac{1}{2}\sin\sqrt{y}}\,dy = 0$

⑤ $\displaystyle\int_0^1\int_0^3 \boxed{xe^{xy}}\,dx\,dy$

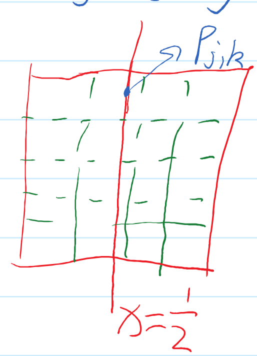
$\displaystyle= \int_0^3\boxed{\int_0^1 xe^{xy}\,dy}\,dx$ [by Fubini's Theorem]

$\displaystyle= \int_0^3 e^{xy}\Big|_0^1\,dx$

$\displaystyle= \int_0^3 [e^x - 1]\,dx = [e^x - x]_0^3 = e^3 - 3 - 1 = e^3 - 4$

⑥ Let f = f(x, y) be a bounded function defined in R = [0, 1] × [0, 1] which is 0 everywhere except along the line x = 1/2.
Show that f is integrable in R with integral equal to 0.

Pf: Let $P = \{(x_j, y_k) : 0 = x_0 < x_1 < \cdots < x_n = 1,\ 0 = y_0 < y_1 < \cdots < y_n = 1\}$ be any partition of R. and $P_{jk} \in [x_{j-1}, x_j] \times [y_{k-1}, y_k]$.



∴ there is a $j_0 \in \{1, 2, \cdots, n\}$ such that

$$\begin{cases} P_{j_0k} = \frac{1}{2} & \text{for each } k \\ \text{or } P_{j_0k} \in (x_{j_0-1}, x_j) \times [y_{k-1}, y_k] \end{cases}$$

Then we compute the Rieman sum

$$|S(f, \mathring{P})| = \left|\sum_{j=1}^{n}\sum_{k=1}^{n} f(P_{jk})\Delta x_j \Delta y_k\right|$$

$$= \left|\sum_{k=1}^{n} f(P_{j_0k})\Delta x_{j_0}\Delta y_k + \sum_{k=1}^{n} f(P_{(j_0+1)k})\Delta x_{j_0+1}\Delta y_k\right|$$

$$\le \sum_{k=1}^{n} |f|_{max}\Delta x_{j_0}\Delta y_k + \sum_{k=1}^{n} |f|_{max}\Delta x_{j_0+1}\Delta y_k$$

$$\leq \sum_{k=1}^{n} |f|_{max} \underbrace{\Delta x_{j_0}}_{\leq |P|} \Delta y_k + \sum_{k=1}^{n} |f|_{max} \underbrace{\Delta x_{j_0+1}}_{|P|} \Delta y_k$$

$$\leq 2 |f|_{max} \cdot |P| \cdot \sum_{k=1}^{n} \Delta y_k$$

$$= 2|f|_{max} \cdot |P| \cdot 1 = 2|f|_{max} \cdot |P|.$$

For any $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{4|f|_{max}} > 0$

then for any partition P with $|P| < \delta$,

$$|S(f, \dot{P})| \leq 2|f|_{max} \cdot |P| \leq 2|f|_{max} \cdot \frac{\varepsilon}{4|f|_{max}} = \frac{\varepsilon}{2} < \varepsilon$$

$\therefore \int_R f = 0$ #.

(7) Corollary: If two bounded functions f, g on $R = [0,1] \times [0,1]$ are the same except at finitely many points and lines, then $\int_R f = \int_R g$

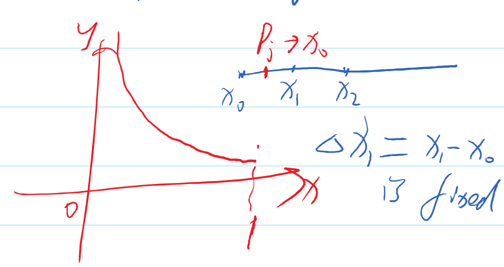
(8) Consider the function ϕ(x) = x^{-a} where a is positive for x ∈ (0, 1] and set ϕ(0) = 1 so that ϕ is a well-defined function on [0, 1]. Show that ϕ is not integrable on [0, 1]. This is the simplest example of an unbounded function. Suggestion: You could use proof by contradiction. Assume it is integrable and then draw a contradiction.

x^{-a} is not Riemann integrable.

Pf: Let $P = \{x_j,\ 0 = x_0 < x_1 < \dots < x_n = 1\}$ be any partition of $[0, 1]$.

Let $P_j \in [x_{j-1}, x_j]$

$$S(\phi, \dot{P}) = \sum_{j=1}^{n} f(P_j) \Delta x_j$$



if $P_1 \to 0$, then $f(P_1) \to \infty$

then $S(\phi, \dot{p}) \to \infty$ as $P_1 \to 0$.

then $S(\phi, \dot{P}) \to \infty$ as $P_1 \to 0$.

⑧ $\int_0^1 x^{-\frac{1}{2}}\,dx$ exists. but $x^{-\frac{1}{2}}$ is not Rieman integrable on $[0,1]$.

$:\stackrel{def}{=} \lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{-\frac{1}{2}}\,dx$ exists because $x^{-\frac{1}{2}}$ is Riemann integrable on $[\varepsilon, 1]$ for any $\varepsilon \in (0,1)$.

⑨ let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

$\int f\,dR = 0$ in the lebsge sense of integration.

the point is $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$.

Let $P = \{x_j : 0 = x_0 < x_1 < \dots < x_n = 1\}$ be any partition of $[0,1]$.

Let $p_j \in (x_{j-1}, x_j) \cap \mathbb{Q}$, $\tilde{p}_j \in (x_{j-1}, x_j) \setminus \mathbb{Q}$

let $\dot{P} = (P, p_j)$, $\tilde{\dot{P}} = (P, \tilde{p}_j)$

$$S(f, \dot{P}) = \sum_{j=1}^{n} f(p_j)\Delta x_j = \sum_{j=1}^{n} 1 \cdot \Delta x_j = 1$$

$$S(f, \tilde{\dot{P}}) = \sum_{j=1}^{n} f(\tilde{p}_j)\Delta x_j = 0$$

$\therefore f$ is not Riemann integrable.